Problem 1

Let $\left\{\frac{1}{\sqrt{2}}, \cos(kx), \sin(kx)\right\}_{k\in\mathbb{N}^+}$

be a basis $\{\varphi_k\}_{k\in\mathbb{N}}$ for $L^2([-\pi,\pi])$

For any $f\in L^2([-\pi,\pi])$, if $c_k = \langle f, \varphi_k\rangle_{L^2([-\pi,\pi])}$

Then $\displaystyle\sum_{k=1}^{n} |c_k|^2 \leq \|f\|^2_{L^2([-\pi,\pi])}$

This is Bessel's inequality, your book states it with $\lim_{n\to\infty}$ or $\sum_{k=1}^{\infty} |c_k| \leq \|f\|^2_{L^2}$

Both answers are acceptable

Proof

Assume V is an n dimensional subspace of $L^2([-\pi,\pi])$ spanned by the 1st n elements of the basis

It follows that if $p_n = \sum_{k=1}^{n} d_k\varphi_k$

$$\|f-p_n\|^2 = \langle f-p_n, f-p_n\rangle$$

$$= \|f\|^2 - 2\langle f, p_n\rangle + \|p_n\|^2$$

$$= \|f\|^2 - 2\sum_{k=1}^{n} c_k d_k + \sum_{k=1}^{n} d_k^2$$

and $\|p_n\|^2 = \sum_{k=1}^{n} d_k \varphi_k$ as φ_k is an onb

It follows that

$$\|f-p_n\|^2 = \|f\|^2 - \underbrace{\sum_{k=1}^{n} |c_k|^2}_{\|s_n\|^2} + \sum_{k=1}^{n} |c_k - d_k|^2$$

where $S_n = \sum_{k=1}^{n} c_k \varphi_k$

In order to minimize the LHS we set $c_k = d_k$.

Then

$$0 \leq \|f-s_n\|^2 = \|f\|^2 - \|s_n\|^2 = \|f\|^2 - \sum_{k=1}^{n} |c_k|^2$$

$$\sum_{k=1}^{n} |c_k|^2 \leq \|f\|^2$$

as the basis is complete taking the limit of the LHS also gives the result as $n \to \infty$

(b) We learned Parsevals equality is

(1) $$\sum_{k=1}^{\infty} |c_k|^2 = \|f\|^2_{L^2([-\pi,\pi])}$$

But there is a discrepancy with your book, which states

(2) $$\langle f, g\rangle = \sum_{k=1}^{\infty} c_k \overline{d_k}$$

with $c_k = \langle f, \varphi_k\rangle$, $d_k = \langle g, \varphi_k\rangle$

Both answers are acceptable

Proof of 1

$$\lim_{n\to\infty} \|f - S_n\|^2 = \|f\|^2 - \lim_{n\to\infty} \sum_{k=1}^{n} |c_k|^2$$

$$= \|f\|^2 - \sum_{k=1}^{\infty} |c_k|^2$$

Completeness can only hold $\Leftrightarrow$ the RHS vanishes giving Plancherel's identity

Proof of 2

Given

$$\langle f, g \rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 \right)$$

we apply Plancherel's formula to each term on the RHS

$$\langle f, g \rangle = \frac{1}{4} \sum_{k=1}^{\infty} \left((c_k + d_k)^2 - (c_k - d_k)^2 \right)$$

$$= \sum_{k=1}^{\infty} c_k d_k$$

Problem 2

$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right) dx$$

$$= \int_{-\infty}^{\infty} u_t u_{tt} + c^2 u_x u_{xt}\, dx = c^2 \int_{-\infty}^{\infty} u_t u_{xx} + u_x u_{xt}\, dx$$

$$= c^2 \int_{-\infty}^{\infty} \frac{d}{dx}\left(u_x u_t\right) dx - a\int_{-\infty}^{\infty} u_t^2\, dx$$

where we used the fact

$$u_{tt} = c^2 u_{xx} + a u_t$$

as $u_x, u_t \to 0$ as $x \to \infty$

$$\frac{dE}{dt} \leq -a \underbrace{\int_{-\infty}^{\infty} u_t^2\, dx}_{\text{positive}}$$

also positive

thus $\frac{dE}{dt}$ is a nonincreasing function of t

Problem 2b

If $u(t,x)$ is a solution with

$$u(t,x) = w_1(t,x) - w_2(t,x)$$

and w_1, w_2 have the same initial data, then

$$u(0,x) = 0 \quad \text{and} \quad E(0) = 0$$

Then $\forall t\; E(t) \leq E(0) \Rightarrow E(t) = 0$

$$\Rightarrow u(t,x) = 0 \quad \forall t$$

Therefore $w_1 = w_2 \quad \forall t, x$

Problem 3

(a) $u(t,x) = v(t,\xi) = v(t, x-t)$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial v}{\partial t} \quad \text{using the PDE} \Rightarrow$$

$$\frac{\partial v}{\partial t} = 0 \quad \text{since} \quad u(1,x) = \frac{x}{1+x^2},$$

$$v(t,\xi) = \frac{x}{1+x^2} \qquad v(t,x-t) = u(t,x)$$

Then the solution $u(t,x) = \dfrac{x+1-t}{1+(x-t+1)^2}$

is the only one that satisfies
$u(1,x) = \dfrac{x}{1+x^2}$ and the PDE

Problem 36

$u_t + 2u_x = 1$

$v(t, x-2t) = u(t, x)$

$\frac{\partial v}{\partial t} = 1$

$v(t, \xi) = t + C(x)$

$v(t, x-2t) = u(t, x)$

$t = 0 \quad v(0, x) = u(0, x) = C(x) = e^{-x^2}$

$v(t, x) = t + e^{-x^2}$

$v(t, x-2t) = u(t, x) = t + e^{-(x-2t)^2}$

Problem 4

a

$$\lim_{n\to\infty} V_n(x) = \lim_{n\to\infty} \frac{2nx}{1+n^2x^2}$$

$$= \lim_{n\to\infty} \frac{2x}{\frac{1}{n}+nx^2} = 0$$

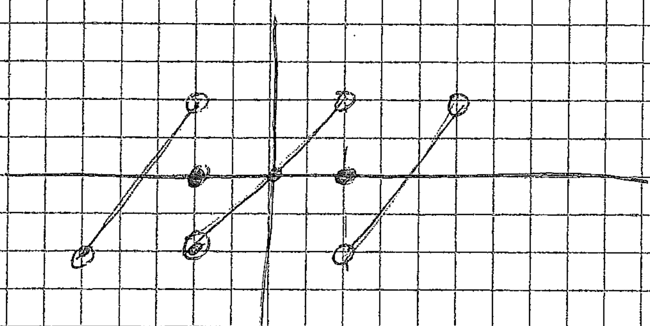
Thus $V_n(x)$ goes pointwise to 0

$$\sup_{x\in\mathbb{R}} |V_n(x)| \geq \frac{2n\left(\frac{1}{n}\right)}{1+\left(\frac{1}{n}\right)^2 n^2} = 1$$

This does not go to 0, so $V_n(x)$ doesn't converge uniformly to zero (5 pts)

b

$f(x) = x$, we need a 2π periodic extension



$$\tilde{f}(x) = \begin{cases} x & -\pi < x < \pi \\ 0 & x = \pm\pi \end{cases}$$

x is odd about 0, so even FS coefficients vanish.

$$b_k = \int_0^{\pi} \frac{2}{\pi} x \sin kx \, dx = -\frac{2}{k} \cos(k\pi)$$

Fourier series $2\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k}$

which converges to $\tilde{f}$

The fourier series of a piecewise C^1 function converges to

$$\frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

where $f(x^+)$ and $f(x^-)$ are the limits from the left + right respectively